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RELATIONS OF THE NONLINEAR THEORY OF THREE-LAYERED SHELLS WITH LAYERS OF VARIABLE THICKNESS

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The versions of the theory of three-layered shells with layers of variable thickness proposed thus far have been based on various physical or kinematic hypotheses, as a rule constructed with a number of constraints on the values of the variables, the thicknesses of the layers and their variation, etc. [1-6]. The diversity of design of such shells made of ordinary materials and composites as well as their service conditions requires that resolvents of a more general form, free of the above-mentioned constraints, be constructed.

In the work reported here we constructed the necessary complex of relations of the geometric nonlinear theory for arbitrary displacements for three-layered shells with an arbitrary geometry of external layers and filler; this complex is based on the static-kinematic model of a broken line, which has been well tested in computational practice [7]. The model used here and the corresponding equations are simplified very much to serve as a base for constructing linearized neutral equilibrium equations and formulating the corresponding problems on determining mixed [8] forms of destabilization of three-layer structural members of the class under consideration with a significant subcritical time of their stress–strain state. In particular, the general equations constructed in our work had to be applied to problems on the analysis of the stress–strain state and on determination of the critical values of acting loads for a number of aircraft structural members which have a considerably varying filler thickness and are subject to transverse bending in use (flaps, ailerons, and slats, made as three-layered plates and shells with layers of variable thickness, tail sections of the main rotor of a helicopter, etc.), as well as some problems of the engineering mechanics of three-layered structural members.

1. Problems associated with parametrizations in the noncanonical regions occupied by the layers are not trivial in the construction of a theory of three-layered shells with layers of variable thickness. These matters were investigated to various extents in [2, 3, 5, 9]. Following those studies, in order to parametrize the middle surfaces $\sigma_{(k)}$ ($k = 1, 2$) of the outer layers of a three-layered shell as the basis for parametrization we choose the middle surface of the filler $\sigma = \sigma_{(3)}$,* assuming that the vector equation $\bar{r} = \bar{r}(\alpha^i)$ is given for it, and that the components (a_{in}, a^{in}) and (b_{in}, b_i^n, b^{in}) of the first and second metric tensors and (c_{in}, c^{in}) of the discriminant tensor, the Christoffel symbols (Γ_{in}^s), and other quantities determining the geometry of σ are specified. Using the method of normal fictitious deformation [9], we parametrize $\sigma_{(k)}$ in two stages, making it possible to solve this problem more correctly than in [2, 3, 5]. In the first stage we map σ onto the coupling (contact) surface of the layers $\sigma_{(kc)}$ by means of the vector equation (see Fig. 1)

$$\bar{r}_{(kc)} = \bar{r} + h_{(k)}\bar{m} \quad (h_{(k)} = \delta_{(k)}h,\ h = t_{(3)}), \tag{1.1}$$

and in the second stage we map surfaces $\sigma_{(kc)}$ onto $\sigma_{(k)}$, determining the radius-vector $\bar{r}_{(k)}$ of points $M_{(k)} \in \sigma_{(k)}$ by

*The index (3), which pertains to parameters of the filler, is henceforth omitted as a rule; indices (k) and (kc) pertain to parameters on $\sigma_{(k)}$ and $\sigma_{(kc)}$, respectively.

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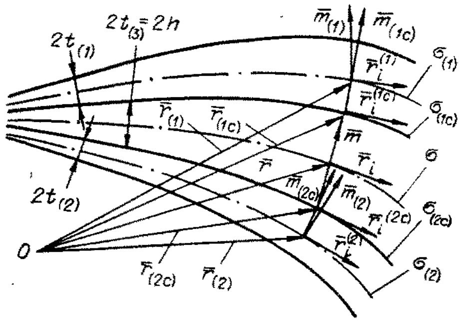

Fig. 1

$$\bar{r}_{(k)} = \bar{r}_{(kc)} + \tilde{t}_{(k)}\bar{m}_{(kc)} \quad (\tilde{t}_{(k)} = \delta_{(k)}t_{(k)}), \tag{1.2}$$

where $\bar{m}$, $\bar{m}_{(kc)}$, and $\bar{m}_{(k)}$ are unit vectors of the normals to σ, $\sigma_{(kc)}$, $\sigma_{(k)}$, respectively; $2t_{(3)}$ and $2t_{(k)}$ are the thicknesses of the filler and the outer layers, measured in the direction of the normals $\bar{m}$ and $\bar{m}_{(k)}$; $\delta_{(1)} = -\delta_{(2)} = 1$. The vector equations (1.1) and (1.2) make it possible to determine on $\sigma_{(kc)}$ and $\sigma_{(k)}$ the basis vectors

$$\begin{gathered}
\bar{r}_i^{(kc)} = \theta^s_{i(kc)}\bar{r}_s + h_i^{(k)}\bar{m}, \quad \bar{m}_{(kc)} = \xi_{(kc)}\left(\bar{m} - h'^s_{(k)}\theta^n_{s(kc)}\bar{r}_n\right),\\
\bar{r}_i^{(k)} = \theta^s_{i(k)}\bar{r}_s^{(kc)} + \tilde{t}_i^{(k)}\bar{m}_{(kc)} = \lambda^s_{i(k)}\bar{r}_s + \lambda_i^{(k)}\bar{m},\\
\bar{m}_{(k)} = \xi_{(k)}\left(\bar{m}_{(kc)} - \tilde{t}'^s_{(k)}\theta^n_{s(k)}\bar{r}_n^{(kc)}\right) = \mu_{(k)}\bar{m} + \mu^s_{(k)}\bar{r}_s.
\end{gathered} \tag{1.3}$$

Equations (1.3) and the corresponding formulas from [9] can be used to find relations for calculating on $\sigma_{(kc)}$ and $\sigma_{(k)}$ the components ($c_{in}^{(kc)}$, $c_{in}^{(k)}$) and ($a_{in}^{(kc)}$, $a_{in}^{(k)}$, $b_{in}^{(kc)}$, $b_{in}^{(k)}$) of the discriminant and metric tensors, respectively, the Christoffel symbols ($\Gamma_{in}^{(kc)s}$, $\Gamma_{in}^{(k)s}$), as well as other quantities that appear in the relations of the theory of shells. In particular,

$$\begin{gathered}
c_{in}^{(kc)} = \xi^{-1}_{(kc)}\theta_{(kc)}c_{in}, \quad c_{in}^{(k)} = \xi^{-1}_{(k)}\theta_{(k)}c_{in}^{(kc)},\\
a_{in}^{(kc)} = \theta^s_{i(kc)}\theta^m_{n(kc)}a_{sm} + h_i^{(k)}h_n^{(k)}, \quad a_{in}^{(k)} = \lambda^s_{i(k)}\lambda^m_{n(k)}a_{sm} + \lambda_i^{(k)}\lambda_n^{(k)},\\
b_{in}^{(kc)} = \xi_{(kc)}\left(b'^s_{i(kc)}a_{sn}^{(kc)} + \nabla_i'^{(kc)}h_n^{(k)}\right), \quad b_{in}^{(k)} = \xi_{(k)}\left(b'^s_{i(k)}a_{sn}^{(k)} + \nabla_i'^{(k)}\tilde{t}_n^{(k)}\right).
\end{gathered} \tag{1.4}$$

Equations (1.3) and (1.4) contain the functions

$$\begin{gathered}
\theta_{(kc)} = 1 - 2h_{(k)}K + h^2_{(k)}\Gamma, \quad \theta_{(k)} = 1 - 2\tilde{t}_{(k)}K_{(kc)} + \tilde{t}^2_{(k)}\Gamma_{(kc)},\\
\xi_{(kc)} = \left(1 + h'^i_{(k)}h_i^{(k)}\right)^{-1/2},\\
\xi_{(k)} = \left(1 + \tilde{t}'^i_{(k)}\tilde{t}_i^{(k)}\right)^{-1/2}, \quad \theta^s_{i(kc)} = \delta^s_i - h_{(k)}b^s_i, \quad \theta^s_{i(k)} = \delta^s_i - \tilde{t}_{(k)}b^s_{i(kc)}.\\
h_i^{(k)} = \partial h_{(k)}/\partial\alpha^i, \quad \tilde{t}_i^{(k)} = \partial\tilde{t}_{(k)}/\partial\alpha^i, \; h'^s_{(k)} = a'^{si}_{(kc)}h_i^{(k)}, \quad \tilde{t}'^i_{(k)} = a'^{is}_{(k)}\tilde{t}_s^{(k)},\\
\lambda_i^{(k)} = \theta^s_{i(k)}h_s^{(k)} + \tilde{t}_i^{(k)}\xi_{(kc)}, \quad \lambda^n_{i(k)} = \theta^s_{i(kc)}\left(\theta^n_{s(k)} - h'^n_{(k)}\tilde{t}_s^{(k)}\xi_{(kc)}\right),\\
\mu_{(k)} = \xi_{(k)}\left(\xi_{(kc)} - \tilde{t}'^s_{(k)}h_n^{(k)}\theta^n_{s(k)}\right), \quad \mu^n_{(k)} = -\xi_{(k)}\theta^n_{s(kc)}\left(\xi_{(kc)}h'^s_{(k)} + \tilde{t}'^m_{(k)}\theta^s_{m(k)}\right),\\
a'^{(k)}_{in} = \theta^s_{i(k)}\theta^k_{n(k)}a^{(kc)}_{sk}, \quad a'^{(kc)}_{in} = \theta^s_{i(kc)}\theta^k_{n(kc)}a_{sk},\\
b'^n_{i(kc)} = \theta^{-1}_{(kc)}\left(b^n_i - \delta^n_i h_{(k)}\Gamma\right), \quad b'^n_{i(k)} = \theta^{-1}_{(k)}\left(b^n_{i(kc)} - \delta^n_i\tilde{t}_{(k)}\Gamma_{(kc)}\right),
\end{gathered} \tag{1.5}$$

where K, $K_{(kc)}$ and Γ, $\Gamma_{(kc)}$ are the average and Gaussian curvatures of surfaces σ and $\sigma_{(kc)}$; and $\nabla'^{(kc)}$ and $\nabla'^{(k)}$ are the operators of covariant differentiation with respect to the metrics defined by $a'^{(kc)}_{in}$ and $a'^{(k)}_{in}$, respectively. The rest of the notation is the same as in [5].

The relations obtained between the geometric parameters on σ, $\sigma_{(kc)}$, and $\sigma_{(k)}$ can be simplified to within $1 + \varepsilon \approx 1$ (ε is a small quantity) for the class of three-layered shells used most commonly in structural members, namely shells whose outer layers are thin and their thicknesses vary little along the coordinates α^i, i.e., the following estimates obtain (L is the characteristic linear dimension of the shell):

$$|2t_{(k)}/L|_{\max} \sim \varepsilon, \quad |2t_i^{(k)}/\sqrt{a_{ii}}|_{\max} \sim \sqrt{\varepsilon}, \quad \varepsilon \ll 1. \tag{1.6}$$

For this class of shells the surfaces $\sigma_{(k)}$, in the sense of [9], slope mildly relative to $\sigma_{(kc)}$ and approximate relations (for $\xi_{(k)} \approx \theta_{(k)} \approx 1$, $\theta^s_{i(k)} \approx \delta^s_i$) can be obtained from relations (1.3)-(1.5) with the assumed accuracy:

$$\bar{r}_i^{(k)} = \bar{r}_i^{(kc)} + \tilde{t}_i^{(k)}\overline{m}_{(kc)} = \theta^n_{i(kc)}\bar{r}_n + H_i^{(k)}\overline{m},$$
$$\overline{m}_{(k)} = \overline{m}_{(kc)} - \tilde{t}_s^{(k)}\bar{r}^s_{(kc)} = \xi_{(kc)}\overline{m} - H'^s_{(k)}\theta^n_{s(kc)}\bar{r}_n, \quad c_{in}^{(k)} = c_{in}^{(kc)},$$
$$a_{in}^{(k)} = a_{in}^{(kc)}, \quad b_{in}^{(k)} = b_{in}^{(kc)} + \nabla_i^{(k)}\tilde{t}_n^{k} \quad (H_i^{(k)} = h_i^{(k)} + \tilde{t}_i^{(k)}\xi_{(kc)}). \tag{1.7}$$

Equations (1.7), in turn, can be reduced to the following form [5, 6] [for $\theta^s_{i(kc)} \approx \delta^s_i$, $\xi_{(kc)} \approx 1$, $H_{(k)} = h_{(k)} + \tilde{t}_{(k)}$] with the same degree of accuracy for thin three-layered shells ($|2H_{(k)}/L|_{\max} \sim \varepsilon$), whose surfaces are mildly sloped relative to σ ($\left(|H_i^{(k)}H_n^{(k)}/\sqrt{a_{ii}a_{nn}}|_{\max} \sim \varepsilon\right)$):

$$\bar{r}_i^{(k)} = \bar{r}_i + H_i^{(k)}\overline{m}, \quad \overline{m}_{(k)} = \overline{m} - H_i^{(k)}\bar{r}^i = \overline{m} - H^i_{(k)}\bar{r}_i,$$
$$a_{in}^{(k)} = a_{in}, \quad b_{in}^{(k)} = b_{in} + \nabla_i H_n^{(k)}. \tag{1.8}$$

Relations (1.3)-(1.8) take into account the differences in the position of the basis vectors on σ and $\sigma_{(k)}$ in the components of the discriminant and metric tensors, the Christoffel symbols, and other quantities that determine the geometry of σ and $\sigma_{(k)}$. Within the framework of the static-kinematic models used for the shell layers it thus becomes possible to include the main geometric features of structures of the given class and the attendant distinctive features of the mechanics of their deformation.

2. When a three-layered stack is considered by using the broken-line model [7], within which the Kirchhoff–Love hypothesis is applied to the outer layers and the hypotheses of the Timoshenko theory in the refined formulation [10] are applied to the filler, the displacement vectors $\overline{V}^z_{(\beta)}$ of points in levels $z_{(\beta)}$ from $\sigma_{(\beta)}$ are written as ($\beta = \overline{1, 3}$)

$$\overline{V}^z_{(\beta)} = \overline{V}_{(\beta)} + z_{(\beta)}\bar{\gamma}_{(\beta)} \quad (-t_{(\beta)} \leqslant z_{(\beta)} \leqslant t_{(\beta)}). \tag{2.1}$$

Here

$$\overline{V}_{(\beta)} = u_i^{(\beta)}\bar{r}^i_{(\beta)} + w_{(\beta)}\overline{m}_{(\beta)}, \quad \bar{\gamma}_{(\beta)} = \gamma_i^{(\beta)}\bar{r}^i_{(\beta)} + \gamma_{(\beta)}\overline{m}_{(\beta)}, \quad \bar{\gamma}_{(k)} = \overline{m}^*_{(k)} - \overline{m}_{(k)} \tag{2.2}$$

are the displacement vectors of points of the middle surface $\sigma_{(\beta)}$ and the rotation vectors of the normals $\overline{m}_{(\beta)}$ to $\sigma_{(\beta)}$ during deformation; $\bar{r}_i^{*(\beta)}$ and $\overline{m}^*_{(\beta)}$ are the coordinate vectors of the main bases, constructed on the deformed surfaces $\sigma^*_{(\beta)}$ [10].

The components of the vectors $\bar{\gamma} = \bar{\gamma}_{(3)}$ for the filler are the desired unknowns while the components of the vectors $\bar{\gamma}_{(k)}$ can be expressed in terms of the components of the displacement vectors $\overline{V}_{(k)}$ of points $\sigma_{(k)}$ and their derivatives with respect to α^i [10].

The representation of the displacement vectors in the layers of a shell by (2.1) and (2.2) correspond to the components of their strain tensors, calculated at levels $z_{(\beta)}$ from $\sigma_{(\beta)}$ from the formulas

$$\varepsilon_{in}^{z(\beta)} = \varepsilon_{in}^{(\beta)} + z_{(\beta)}\chi_{in}^{(\beta)}, \quad 2\varepsilon_{i3}^{z(3)} = 2\varepsilon_{i3} + z_{(3)}\nabla_i\varepsilon_3, \quad \varepsilon_{33}^{z(3)} = \varepsilon_3, \quad \varepsilon_{i3}^{z(k)} = \varepsilon_{33}^{z(k)} = 0, \tag{2.3}$$

where

$$2\varepsilon_{in}^{(\beta)} = a_{in}^{*(\beta)} - a_{in}^{(\beta)}, \quad \chi_{in}^{(k)} = b_{in}^{(k)} - b_{in}^{*(k)},$$
$$2\chi_{in}^{(3)} = \bar{r}_i^*\nabla_n\bar{\rho}_3^* + \bar{r}_n^*\nabla_i\bar{\rho}_3^* + 2b_{in}, \quad 2\varepsilon_{i3} = \bar{r}_i^*\bar{\rho}_3^*,$$
$$2\varepsilon_3 = \bar{\gamma}(2\overline{m} + \bar{\gamma}), \quad \nabla_i\varepsilon_3 = \bar{\rho}_3^*\nabla_i\bar{\rho}_3^*$$
$$\left(a_{in}^{*(\beta)} = \bar{r}_i^{*(\beta)}\bar{r}_n^{*(\beta)}, \quad b_{in}^{*(\beta)} = -\bar{r}_i^{*(\beta)}\overline{m}_n^{*(\beta)}, \quad \bar{\rho}_3^* = \overline{m} + \bar{\gamma}\right). \tag{2.4}$$

Writing the layer coupling conditions in terms of the displacements

$$\overline{V}^z_{(3)}(z_{(3)} = h_{(k)}) = \overline{V}^z_{(k)}\left(z_{(k)} = -\tilde{t}_{(k)}\right) \quad \text{or} \quad \overline{V}_{(kc)} = \overline{V}_{(3k)} \quad (k = 1, 2)$$

and using Eqs. (1.3), (1.5), (2.1), and (2.2), we find the relations between the displacement vectors in the layers of the shell

$$\overline{V}_{(k)} = \overline{V}_{(3k)} + \tilde{t}_{(k)}\bar{\gamma}_{(k)} \tag{2.5}$$

and their components

$$u_i^{(k)} = \lambda^s_{i(k)}u_s^{(3k)} + \lambda_i^{(k)}w_{(3k)} + \tilde{t}_{(k)}\gamma_i^{(k)}, \quad w_{(k)} = \mu_{(k)}w_{(3k)} + \mu^s_{(k)}u_s^{(3k)} + \tilde{t}_{(k)}\gamma_{(k)}.$$

Here we have introduced notation for the displacement vectors of points of the coupling surface $\sigma_{(kc)}$ in the outer layers $\overline{V}_{(kc)}$ and the filler $\overline{V}_{(3k)}$, lying along basis vectors constructed on $\sigma_{(k)}$ and σ:

$$\overline{V}_{(kc)} = \overline{V}_{(k)} - \tilde{t}_{(k)}\overline{\gamma}_{(k)} = u_i^{(kc)}\overline{r}_{(k)}^i + w_{(kc)}\overline{m}_{(k)},$$
$$\overline{V}_{(3k)} = \overline{V} + h_{(k)}\overline{\gamma} = u_i^{(3k)}\overline{r}^i + w_{(3k)}\overline{m}. \tag{2.6}$$

The components of the vectors in (2.6) are expressed in terms of the previously introduced components (2.2) by means of

$$u_s^{(kc)} = u_s^{(k)} - \tilde{t}_{(k)}\gamma_s^{(k)}, \quad w_{(kc)} = w_{(k)} - \tilde{t}_{(k)}\gamma_{(k)}, \quad u_s^{(3k)} = u_s + h_{(k)}\gamma_s,$$
$$w_{(3k)} = w + h_{(k)}\gamma.$$

Using Eqs. (1.1)-(1.4), which parametrize the surfaces $\sigma_{(kc)}$ and $\sigma_{(k)}$ relative to σ in the initial state, and the relations found between the displacement vectors in the layers of the shell (2.5), we determine the radius-vectors of the points $M^*_{(kc)} \in \sigma^*_{(kc)}$ and $M^*_{(k)} \in \sigma^*_{(k)}$ in the deformed state:

$$\overline{r}^*_{(kc)} = \overline{r}_{(kc)} + \overline{V}_{(kc)} = \overline{r}_* + h_{(k)}\overline{\rho}_3^*, \quad \overline{r}^*_{(k)} = \overline{r}_{(k)} + \overline{V}_{(k)} = \overline{r}^*_{(kc)} + \tilde{t}_{(k)}\overline{m}^*_{(k)} + \tilde{t}_{(k)}\overline{\psi}_{(k)} \tag{2.7}$$

as well as the basis vectors constructed on $\sigma^*_{(kc)}$ and $\sigma^*_{(k)}$:

$$\overline{r}_i^{*(kc)} = \overline{r}_i^* + h_i^{(k)}\overline{\rho}_3^* + h_{(k)}\nabla_i\overline{\rho}_3^*, \quad \tilde{\theta}^n_{i(k)}\overline{r}_n^{*(k)} = \overline{r}_i^{*(kc)} + \tilde{t}_i^{(k)}(\overline{m}^*_{(k)} + \overline{\psi}_{(k)}) + \tilde{t}_{(k)}\nabla_i\overline{\psi}_{(k)} . \tag{2.8}$$

Here

$$\overline{r}_* = \overline{r} + \overline{V}, \quad \overline{\psi}_{(k)} = \overline{m}_{(kc)} - \overline{m}_{(k)}, \quad \tilde{\theta}^n_{i(k)} = \delta_i^n + \tilde{t}_{(k)}b^{*n}_{i(k)}, \quad \overline{V} = \overline{V}_{(3)}. \tag{2.9}$$

If the thicknesses of the outer layers vary little along the coordinates α^i that satisfy conditions (1.6), then the last equations in (2.7) and (2.8) can be rewritten to within $1 + \varepsilon \approx 1$ in the approximate form

$$\overline{r}^*_{(kc)} = \overline{r}^*_{(k)} - \tilde{t}_{(k)}\overline{m}^*_{(k)}, \quad \overline{r}_i^{*(kc)} = \tilde{\theta}^n_{i(k)}\overline{r}_n^{*(k)} - \tilde{t}_i^{(k)}\overline{m}^*_{(k)}. \tag{2.10}$$

Equations (1.3) and (1.7), (2.8) and (2.10) make it possible to find the relations between the components of the tangential strain tensors of surfaces $\sigma_{(kc)}$ and σ:

$$2\varepsilon_{in}^{(kc)} = a_{in}^{*(kc)} - a_{in}^{(kc)} = 2\varepsilon_{in}^{(3)} + 2h_{(k)}\chi_{in}^{(3)} + h_i^{(k)}(2\varepsilon_{n3} + h_{(k)}\nabla_n\varepsilon_3) + h_n^{(k)}(2\varepsilon_{i3} + h_{(k)}\nabla_i\varepsilon_3) + 2h_i^{(k)}h_n^{(k)}\varepsilon_3 + h^2_{(k)}\nu_{in}, \tag{2.11}$$

as well as surfaces $\sigma_{(kc)}$ and $\sigma_{(k)}$:

$$\varepsilon_{in}^{(kc)} = \varepsilon_{in}^{(k)} - \tilde{t}_{(k)}\chi_{in}^{(k)} + \tilde{t}^2_{(k)}\nu_{in}^{(k)}/2, \tag{2.12}$$

which admit further simplifications since the last terms containing $h^2_{(k)}$ and $\tilde{t}^2_{(k)}$ can be ignored. Here the components of the strains in the layers of the shell are calculated from (2.3) and (2.4), and

$$\nu_{in} = \nabla_i\overline{\rho}_3^*\nabla_n\overline{\rho}_3^* - \nabla_i\overline{m}\nabla_n\overline{m}, \quad \nu_{in}^{(k)} = b^{*s}_{i(k)}b^{*(k)}_{ns} - b^s_{i(k)}b^{(k)}_{ns}.$$

To determine the relations between the components of the bending strain tensors of the surfaces under consideration, we construct the normal unit vectors $\overline{m}_*$ to σ_* and $\overline{m}^*_{(kc)}$ to $\sigma^*_{(kc)}$, writing them as the expansions

$$\overline{m}_* = \xi_*(\overline{\rho}_3^* + \xi_i^*\overline{r}_*^i), \quad \overline{m}^*_{(kc)} = \xi^*_{(kc)}(\overline{\rho}_3^* + \xi_i^{*(kc)}\overline{r}^i_{*(kc)}) = \xi^*_{(k)}(\overline{m}^*_{(k)} + \xi_i^{*(k)}\overline{r}^i_{*(k)}). \tag{2.13}$$

From the conditions $\overline{m}_*^2 = \overline{m}^{*2}_{(kc)} = 1$ and $\overline{m}_*\overline{r}_i^* = \overline{m}^*_{(kc)}\overline{r}_i^{*(kc)} = 0$, using (2.4) and (2.8)-(2.10) we find the coefficients of the expansions in (2.13):

$$\xi_i^* = -\overline{\rho}_3^*\overline{r}_i^* = -2\varepsilon_{i3}, \quad \xi_* = (1 + 2\varepsilon_*)^{-1/2}, \quad \xi_i^{*(kc)} = -\overline{\rho}_3^*\overline{r}_i^{*(kc)} = -2\varepsilon_{i3}^{(kc)},$$
$$\xi^*_{(kc)} = (1 + 2\varepsilon^*_{kc})^{-1/2}, \quad \xi_i^{*(k)} = \tilde{t}_s^{(k)}(\delta_i^s - \tilde{t}_{(k)}b^{*s}_{i(k)}), \quad \xi^*_{(k)} = (1 + \xi^i_{*(k)}\xi_i^{*(k)})^{-1/2},$$
$$\varepsilon_* = \varepsilon_3 - 2\varepsilon^{i3}\varepsilon_{i3}, \quad \varepsilon^*_{(kc)} = \varepsilon_3 - 2\varepsilon^{i3}_{(kc)}\varepsilon_{i3}^{(kc)}, \quad 2\varepsilon_{i3}^{(kc)} = 2\varepsilon_{i3} + h_i^{(k)}(1 + 2\varepsilon_3) + h_{(k)}\nabla_i\varepsilon_3.$$

Differentiating (2.13) with respect to α^i, with allowance for the approximated equations $\xi_* \approx 1 - \varepsilon_*$, $\xi^*_{(kc)} \approx 1 - \varepsilon^*_{(kc)}$, $\xi^*_{(k)} \approx 1$ we determine the components of the second metric tensor of the surfaces σ_*, $\sigma^*_{(kc)}$, and $\sigma^*_{(k)}$

$$b^*_{in} = (1 - \varepsilon_*)(b_{in} + \nabla_i^*\varepsilon_{n3} + \nabla_n^*\varepsilon_{i3} - \chi_{in}^{(3)}), \quad b_{in}^{*(kc)} = b_{in}^{*(k)} - \nabla_i^{*(k)}\tilde{t}_n^{(k)},$$
$$b_{in}^{*(kc)} = (1 - \varepsilon^*_{(kc)})(b_{in} - \chi_{in}^{(3)} + \nabla_i^{*(kc)}\varepsilon_{n3}^{(kc)} + \nabla_n^{*(kc)}\varepsilon_{i3}^{(kc)} - h_i^{(k)}\nabla_n\varepsilon_3 - h_{(k)}\nu_{in})$$

and the components of the bending strain tensors

$$\chi_{in}^{(3)} = \chi_{in}^{0(3)} + \nabla_i^* \varepsilon_{n3} + \nabla_n^* \varepsilon_{i3} - \varepsilon_* b_{in}^*, \quad \chi_{in}^{(k)} = \chi_{in}^{(kc)} + A_{in}^{s(k)} \tilde{t}_s^{(k)},$$
$$\chi_{in}^{(kc)} = \chi_{in}^{(3)} + \varepsilon_{(kc)}^* b_{in}^{*(kc)} + \varepsilon_{(kc)} b_{in}^{(kc)} + h_i^{(k)} \nabla_n \varepsilon_3 + A_{in}^{s(kc)} h_s^{(k)} - \nabla_i^{*(kc)} \varepsilon_{n3}^{(kc)} - \nabla_n^{*(kc)} \varepsilon_{i3}^{(kc)}$$
$$\left(\chi_{in}^{0(3)} = b_{in} - b_{in}^*, \quad A_{in}^{s(k)} = \Gamma_{in}^{(k)*s} - \Gamma_{in}^{(k)s}, \quad A_{in}^{s(kc)} = \Gamma_{in}^{(kc)*s} - \Gamma_{in}^{(kc)s}\right),$$

where ∇_i^*, $\nabla_i^{*(kc)}$, and $\nabla_i^{*(k)}$ are the operators of covariant differentiation with respect to the metrics determined by tensors a_{in}^*, $a_{in}^{*(kc)}$, and $a_{in}^{*(k)}$, respectively.

3. We obtain the equilibrium equations and the corresponding natural boundary conditions by using the Lagrange variational equation

$$\delta\Pi = \delta A - \delta W = 0. \tag{3.1}$$

The variations of the strain energy of the outer layers $\delta W_{(k)}$ and the filler $\delta W_{(3)}$, corresponding to the static-kinematic models adopted, have the form

$$\delta W_{(k)} = \iint\limits_{\sigma_{(k)}} \left(\overline{T}_{(k)}^n \nabla_n \delta \overline{V}_{(k)} + \overline{M}_{(k)}^n \nabla_n \delta \overline{m}_{(k)}^*\right) d\sigma_{(k)},$$
$$\delta W_{(3)} = \iint\limits_{\sigma} \left(\overline{T}_{(3)}^n \nabla_n \delta \overline{V} + \overline{M}_{(3)}^n \nabla_n \delta \overline{\gamma} + \overline{N}_{(3)} \delta \overline{\gamma}\right) d\sigma. \tag{3.2}$$

Here we have introduced the vectors of internal forces and moments in the layers of the shell, per units of length of the coordinate lines of the undeformed surface $\sigma_{(\beta)}$ and given relative to the basis vectors on the deformed surfaces $\sigma_{(\beta)}^*$ ($\beta = \overline{1, 3}$):

$$\overline{T}_{(k)}^n = T_{(k)}^{ns} \overline{r}_s^{*(k)} + M_{(k)}^{ns} \nabla_s \overline{m}_{(k)}^*, \quad \overline{M}_{(k)}^n = M_{(k)}^{ns} \overline{r}_s^{*(k)} + H_{(k)}^{ns} \nabla_s \overline{m}_{(k)}^*,$$
$$\overline{T}_{(3)}^n = T_{(3)}^{ns} \overline{r}_s^* + T_{(3)}^{n3} \overline{\rho}_3^* + M_{(3)}^{ns} \nabla_s \overline{\rho}_3^*, \quad \overline{M}_{(3)}^n = M_{(3)}^{ns} \overline{r}_s^* + M_{(3)}^{n3} \overline{\rho}_3^* + H_{(3)}^{ns} \nabla_s \overline{\rho}_3^*,$$
$$\overline{N}_{(3)} = T_{(3)}^{n3} \overline{r}_n^* + T_{(3)}^{33} \overline{\rho}_3^* + M_{(3)}^{n3} \nabla_n \overline{\rho}_3^*.$$

Their components are calculated from (ν, $\gamma = \overline{1, 3}$)

$$T_{(\beta)}^{\nu\gamma} = \Gamma_{(\beta)}\left(I_{(\beta)} \sigma_{(\beta)}^{\nu\gamma}\right), \quad M_{(\beta)}^{\nu\gamma} = \Gamma_{(\beta)}\left(I_{(\beta)} \sigma_{(\beta)}^{\nu\gamma} z_{(\beta)}\right), \quad H_{(\beta)}^{\nu\gamma} = \Gamma_{(\beta)}\left(I_{(\beta)} \sigma_{(\beta)}^{\nu\gamma} z_{(\beta)}^2\right),$$
$$\Gamma_{(\beta)}(\ldots) = \int\limits_{-t(\beta)}^{t(\beta)} (\ldots)\, dz_{(\beta)}, \quad I_{(\beta)} = \sqrt{g_{(\beta)}/a_{(\beta)}}, \quad g_{(\beta)} = \det\left(\overline{\rho}_i^{(\beta)} \overline{\rho}_n^{(\beta)}\right),$$
$$\overline{\rho}_i^{(\beta)} = \overline{r}_i^{(\beta)} + z_{(\beta)} \overline{m}_i^{(\beta)}.$$

Carrying out the summation of Eqs. (3.2) and taking (2.11)-(2.13) into account, after some transformations we arrive at the following expression for the variation of the strain energy of a three-layered shell $\left(\sum = \sum\limits_{k=1}^{2}\right)$:

$$\delta W = \sum \iint\limits_{\sigma_{(k)}} \widetilde{M}_{(k)}^{in} \overline{r}_i^{*(k)} \nabla_n \delta \overline{m}_{(k)}^*\, d\sigma_{(k)} + \iint\limits_{\sigma} \left(\overline{T}^n \nabla_n \delta \overline{V} + \overline{M}^n \nabla_n \delta \overline{\gamma} + \overline{N} \delta \overline{\gamma}\right) d\sigma, \tag{3.2'}$$

where the vectors of the internal forces and moments for the shell as a whole, reduced to the middle surface σ of the filler, are given by

$$\overline{T}^n = T^{ns} \overline{r}_s^* + T^{n3} \overline{\rho}_3^* + M^{ns} \nabla_s \overline{\rho}_3^*, \quad \overline{M}^n = M^{ns} \overline{r}_s^* + M^{n3} \overline{\rho}_3^* + H^{ns} \nabla_s \overline{\rho}_3^*,$$
$$\overline{N} = T^{n3} \overline{r}_n^* + T^{33} \overline{\rho}_3^* + M^{n3} \nabla_n \overline{\rho}_3^*, \tag{3.3}$$

and their components are $\left(\widetilde{T}_{(k)}^{in} = T_{(k)}^{is} \theta_{s(k)}^{*n} \sqrt{a_{(k)}/a}\right)$

$$T^{ns} = T_{(3)}^{ns} + \sum \widetilde{T}_{(k)}^{ns}, \quad M^{ns} = M_{(3)}^{ns} + \sum \widetilde{T}_{(k)}^{ns} h_{(k)},$$
$$M^{n3} = M_{(3)}^{n3} + \sum \widetilde{T}_{(k)}^{ns} h_{(k)} h_s^{(k)},$$
$$T^{n3} = T_{(3)}^{n3} + \sum \widetilde{T}_{(k)}^{ns} h_s^{(k)}, \quad H^{ns} = H_{(3)}^{ns} + \sum \widetilde{T}_{(k)}^{ns} h_{(k)}^2, \quad T^{33} = T_{(3)}^{33} + \sum \widetilde{T}_{(k)}^{ns} h_n^{(k)} h_s^{(k)}. \tag{3.4}$$

Introducing the contour bending moments and torques of the internal forces $\widetilde{M}_n^{(k)} = \widetilde{M}_{(k)}^{is} n_i^{*(k)} n_s^{*(k)}$, and using formulas for the transformation of surface integrals [10], we get the final expression for δW from Eq. (3.2)', namely,

$$\delta W = -\left(\overline{\overline{M}}_{n\tau} \delta \overline{V} + \overline{\overline{G}}_{n\tau} \delta \overline{\gamma}\right)\Big|_C + \int\limits_C \left[\left(d\overline{\overline{M}}_{n\tau}/dS + \overline{\overline{T}}^i n_i\right) \delta \overline{V} + \right.$$

$$+(\bar{G}_{n\tau}+\overline{\overline{M}}^{i}n_i)\,\delta\bar{\gamma}+\sum\overline{\overline{M}}_n^{(k)}\delta\bar{m}^*_{(k)}]\,dS+\iint\limits_{\sigma}[\nabla_i\overline{\overline{T}}^i\delta\bar{V}+(\nabla_i\overline{\overline{M}}^i-\overline{\overline{N}})\,\delta\bar{\gamma}]\,d\sigma. \tag{3.5}$$

Here

$$\overline{\overline{M}}_{n\tau}=\sum\overline{\overline{M}}_{n\tau}^{(k)};\ \bar{\bar{G}}_{n\tau}=\sum\overline{\overline{M}}_{n\tau}^{(k)}h_{(k)};\ \overline{\overline{M}}_{n\tau}^{(k)}=\widetilde{M}_{n\tau}^{(k)}\theta_{,s}^{(k)}\bar{m}^*_{(k)};\ \widetilde{M}_{(k)}^{is}=M_{(k)}^{is}+T_{(k)}^{is}\tilde{t}_{(k)};$$
$$\overline{\overline{M}}_n^{(k)}=\bar{n}^*_{(k)}\,(\widetilde{M}_n^{(k)}+\widetilde{M}_{n\tau}^{(k)}\tilde{t}_{(k)}k_{n\tau}^{*(k)})\,ds_{(k)}/ds;\ \ \bar{G}_{n\tau}=\sum h_{(k)}d\overline{\overline{M}}_{n\tau}^{(k)}/ds;$$
$$\theta_{,s}^{(k)}=1+\tilde{t}_{(k)}k_\tau^{*(k)};\ \ \overline{\overline{T}}^i=\bar{T}^i+\sum\bar{N}^i_{(k)}=\widetilde{T}^{is}\bar{r}^*_s+\widetilde{T}^{i3}\bar{m}_*;$$
$$\overline{\overline{M}}^i=\bar{M}^i+\sum\bar{N}^i_{(k)}h_{(k)}=\widetilde{M}^{is}\bar{r}^*_s+\widetilde{M}^{i3}\bar{m}_*;\ \ \overline{\overline{N}}=\bar{N}+\sum\bar{N}^i_{(k)}h_i^{(k)}=\widetilde{N}^i\bar{r}^*_i+\widetilde{N}^3\bar{m}_*;$$
$$\bar{N}^i_{(k)}=\nabla_s^{*(k)}\widetilde{M}^{js}_{(k)}\theta^{*i}_{j(k)}\sqrt{a_{(k)}/a}\,\bar{m}^*_{(k)};\ \ \theta^{*i}_{s(k)}=\delta^i_s-\tilde{t}_{(k)}b^{*i}_{s(k)};$$

$(\bar{n}^*_{(\beta)},\ \bar{\tau}^*_{(\beta)},\ \bar{m}^*_{(\beta)})$, $(\bar{n}_{(\beta)},\ \bar{\tau}_{(\beta)},\ \bar{m}_{(\beta)})$ are right-hand trihedrals of orthogonal unit vectors, constructed on the conour lines $C^*_{(\beta)}\in\sigma^*_{(\beta)}$ and $C_{(\beta)}\in\sigma_{(\beta)}$, respectively. For them we have the representations

$$\bar{n}^*_{(\beta)}=n_i^{*(\beta)}\bar{r}^i_{*(\beta)}=n^i_{*(\beta)}\bar{r}_i^{*(\beta)},\ \ \bar{\tau}^*_{(\beta)}=\tau_i^{*(\beta)}\bar{r}^i_{*(\beta)}=\tau^i_{*(\beta)}\bar{r}_i^{*(\beta)},\ \ \bar{n}_{(\beta)}=n_i^{(\beta)}\bar{r}^i_{(\beta)},$$
$$\bar{\tau}_{(\beta)}=\tau_i^{(\beta)}\bar{r}^i_{(\beta)},$$

where $dS_{(\beta)}$ is an element of arc $C_{(\beta)}\in\sigma_{(\beta)}$, $k_{n\tau}^{*(\beta)}=-b_{is}^{*(\beta)}n^i_{*(\beta)}\tau^s_{*(\beta)}$, $k_\tau^{*(\beta)}=-b_{is}^{*(\beta)}\tau^i_{*(\beta)}\tau^s_{*(\beta)}$.

To find the elementary work of external forces on possible displacements of the shell we assume that the load acting on it is reduced to the vectors of the surface and contour forces $\bar{X}_{(\beta)}$ and $\bar{\Phi}_{(\beta)}$ and moments $\bar{L}_{(\beta)}$, $\bar{H}_{(\beta)}$ per unit deformed area $\sigma_{(\beta)}$ and length $C_{(\beta)}$:

$$\begin{aligned}\bar{X}_{(\beta)}&=X^i_{(\beta)}\bar{r}_i^{*(\beta)}+X^3_{(\beta)}\bar{m}^*_{(\beta)},\quad \bar{\Phi}_{(\beta)}=\Phi_n^{(\beta)}\bar{n}^*_{(\beta)}+\Phi_\tau^{(\beta)}\bar{\tau}^*_{(\beta)}+\Phi_m^{(\beta)}\bar{m}^*_{(\beta)},\\ \bar{L}_{(\beta)}&=L^i_{(\beta)}\bar{r}_i^{*(\beta)}+L^3_{(\beta)}\bar{m}^*_{(\beta)},\quad \bar{H}_{(\beta)}=H_n^{(\beta)}\bar{n}^*_{(\beta)}+H_\tau^{(\beta)}\bar{\tau}^*_{(\beta)}+H_m^{(\beta)}\bar{m}^*_{(\beta)}.\end{aligned} \tag{3.6}$$

For the elementary work of external loads given by vectors (3.6) we write the expressions

$$\delta A=\sum_{\beta=1}^{3}\left[\iint\limits_{\sigma(\beta)}(\bar{X}_{(\beta)}\delta\bar{V}_{(\beta)}+\bar{L}_{(\beta)}\delta\bar{\gamma}_{(\beta)})\,d\sigma_{(\beta)}+\int\limits_{C_{(\beta)}}(\bar{\Phi}_{(\beta)}\delta\bar{V}_{(\beta)}+\bar{H}_{(\beta)}\delta\bar{\gamma}_{(\beta)})\,dS_{(\beta)}\right],$$

and by analogy with (3.2)' and (3.5) recast them in the form

$$\delta A=-(\bar{H}_{n\tau}\delta\bar{V}+\bar{\bar{H}}_{n\tau}\delta\bar{\gamma})\,|_C+\int\limits_C(\bar{\Phi}\delta\bar{V}+\bar{H}\delta\bar{\gamma}+\sum\bar{H}_n^{(k)}\delta\bar{m}^*_{(k)})\,dS+\iint\limits_\sigma(\bar{X}\delta\bar{V}+\bar{L}\delta\bar{\gamma})\,d\sigma. \tag{3.7}$$

Here we introduce the following notation for the contour and surface external forces and moments, reduced to σ:

$$\bar{H}_{n\tau}=\sum\bar{H}_{n\tau}^{(k)};\ \ \bar{\bar{H}}_{n\tau}=\sum\bar{H}_{n\tau}^{(k)}h_{(k)};\ \ \bar{H}_{n\tau}^{(k)}=(H_\tau^{(k)}+\Phi_\tau^{(k)}\tilde{t}_{(k)})\,\theta_{,s}^{(k)}\bar{m}^*_{(k)};$$
$$\bar{H}_n^{(k)}=\bar{n}^*_{(k)}\,[H_n^{(k)}+\Phi_n^{(k)}\tilde{t}_{(k)}+\tilde{t}_{(k)}k_{n\tau}^{*(k)}(H_\tau^{(k)}+\Phi_\tau^{(k)}\tilde{t}_{(k)})]\,dS_{(k)}/dS;$$
$$\bar{\Phi}=\bar{\Phi}_{(3)}+\Sigma[(\bar{\Phi}_{(k)}-\bar{H}_H^{(k)})dS_{(k)}/dS+d\bar{H}_{n\tau}/dS];\ \bar{H}_H^{(k)}=\theta^{*s}_{i(k)}L^i_{(k)}n_s^{(k)}\bar{m}^*_{(k)};$$
$$\bar{H}=\bar{H}_{(3)}+\sum h_{(k)}\,[(\bar{\Phi}_{(k)}-\bar{H}_H^{(k)})\,dS_{(k)}/dS+d\bar{H}_{n\tau}/dS];\ \ \ \bar{H}^s_{(k)}=\theta^{*s}_{i(k)}L^i_{(k)}\bar{m}^*_{(k)};$$
$$\bar{X}=\bar{X}_{(3)}+\sum(\bar{X}_{(k)}+\nabla_s^{*(k)}\bar{H}^s_{(k)})\sqrt{a_{(k)}/a}=X^i\bar{r}^*_i+X^3\bar{m}_*;$$
$$\bar{L}=\bar{L}_{(3)}+\sum h_{(k)}(\bar{X}_{(k)}+\nabla_s^{*(k)}\bar{H}^s_{(k)})\sqrt{a_{(k)}/a}=L^i\bar{r}^*_i+L^3\bar{m}_*.$$

Substituting (3.5) and (3.7) into (3.1), we arrive at the Lagrange variational equation

$$\delta\Pi=(\bar{H}_\tau\delta\bar{V}+\bar{G}_\tau\delta\bar{\gamma})\,|_C+\int\limits_C(\bar{R}\delta\bar{V}+\bar{M}_H\delta\bar{\gamma}+\sum\bar{G}_H^{(k)}\delta\bar{m}^*_{(k)})\,dS+\iint\limits_\sigma(\bar{F}\delta\bar{V}+\bar{Z}\delta\bar{\gamma})\,d\sigma=0, \tag{3.8}$$

from which, by virtue of the arbitrary nature of the variable vector functions, we can obtain equilibrium equations in vectorial form

$$\bar{F}=F^i\bar{r}^*_i+F^3\bar{m}_*=0,\ \ \bar{Z}=Z^i\bar{r}^*_i+Z^3\bar{m}_*=0 \tag{3.9}$$

and scalar form

$$F^i=0,\ F^3=0,\ Z^i=0,\ Z^3=0, \tag{3.10}$$

and the natural boundary conditions on the contour of the shell C

$$\overline{R}=R_n\overline{n}_*+R_\tau\overline{\tau}_*+R_m\overline{m}_*=0 \text{ for } \delta\overline{V}\neq 0,\ \overline{M}_H=M_n^{H}\overline{n}_*+M_\tau^{H}\overline{\tau}_*+ \\ +M_m^{H}\overline{m}_*=0 \quad \text{for } \delta\overline{\gamma}\neq 0,\ \ \sum G_H^{(k)}=0 \text{ for } \overline{n}_{(k)}^*\delta\overline{m}_{(k)}^*\neq 0 \tag{3.11}$$

and its sharp corners

$$\overline{H}_\tau=0 \ \text{ for } \delta\overline{V}\neq 0,\ \overline{G}_\tau=0 \text{ for } \delta\overline{\gamma}\neq 0. \tag{3.12}$$

The following notation has been introduced into Eqs. (3.8)-(3.12):

$$F^i=\nabla_s^*\widetilde{T}^{is}-\widetilde{T}^{s3}b_s^{*i}+X^i;\quad F^3=\nabla_s\widetilde{T}^{s3}+\widetilde{T}^{is}b_{is}^*+X^3;$$
$$Z^i=\nabla_s^*\widetilde{M}^{is}-\widetilde{M}^{s3}b_s^{*i}-\widetilde{N}^i+L^i;$$
$$Z^3=\nabla_s\widetilde{M}^{s3}+\widetilde{M}^{is}b_{is}^*-\widetilde{N}^3+L^3;\quad \overline{R}=\overline{\Phi}-\overline{\overline{T}}^s n_s-d\overline{\overline{M}}_{n\tau}/dS;$$
$$\overline{H}_\tau=\overline{\overline{M}}_{n\tau}-\overline{H}_{n\tau};\quad \overline{M}_H=\overline{H}-\overline{G}_{n\tau}-\overline{M}^i n^i;\ \ \overline{G}_\tau=\overline{\overline{G}}_{n\tau}-\overline{\overline{H}}_{n\tau};$$
$$\overline{G}_H^{(k)}=G_H^{(k)}\overline{n}_{(k)}^*=\overline{H}_n^{(k)}-\overline{\overline{M}}_n^{(k)}.$$

The variational equation (3.8) and the resulting equilibrium equations (3.9), (3.10) and boundary conditions (3.11), (3.12) of the theory of three-layered shells with layers of variable thickness are, within the framework of the static-kinematic model, the most general and valid for arbitrary displacements. The introduction of some limitations on the magnitude of the displacements, the layer thickness, and their variation along α^i makes it possible to simplify the main relations of the proposed version of the theory of three-layered shells and reduce them to the well-known version of the theory. In particular, Eqs. (1.8) are valid for parametrization of the outer layers in a study of the average bend [5, 10] of thin shells, the components of the displacement vector $\overline{V}_{(k)}$ of surface $\sigma_{(k)}$ can be expressed in terms of the components of $\overline{V}$ and $\overline{\gamma}$ of surface σ by means of the given formulas [6]

$$u_i^{(k)}=u_i+h_{(k)}\gamma_i+H_i^{(k)}w-\widetilde{t}_{(k)}\omega_i,\quad w_{(k)}=w+h_{(k)}\gamma-a^{is}H_s^{(k)}u_i^{(k)},$$

by virtue of which $\omega_i^{(k)}\approx\omega_i\left(\omega_i^{(\beta)}=\overline{m}_{(\beta)}\nabla_i\overline{V}_{(\beta)}\right)$. For the given class of three-layered shells the equilibrium equations (3.10) and the boundary conditions (3.11), (3.12), adjusted to the coordinate vectors of the undeformed basis on σ, related to the basis $(\overline{r}_i^*,\ \overline{m}_*)$ by

$$\overline{r}_i^*=\overline{r}_i+\omega_i\overline{m},\ \ \overline{m}_*=\overline{m}-\omega_i\overline{r}^i=\overline{m}-\omega_\tau\overline{\tau}-\omega_n\overline{n},\ \ \overline{n}_*=\overline{n}+\omega_n\overline{m},$$
$$\overline{\tau}_*=\overline{\tau}+\omega_\tau\overline{m},\quad \omega_n=\omega_i n^i,\quad \omega_\tau=\omega_i\tau^i,$$

can be simplified considerably to a form that agrees with [6] if the moments $\widetilde{M}^{s3}$ and H^{in} in (3.3) and (3.4) are ignored. The sixth moment equilibrium equation $Z^3 = 0$ is algebraic in this case.

For thin three-layered shells with very thin outer layers ($2h/L\sim\varepsilon$, $t_{(k)}/h_{(k)}\sim\varepsilon$) the system of equilibrium equations is simplified even more and can be reduced to a form [5] which agrees as to structure of differential operators with the equilibrium equations of the Timoshenko type of theory of single-layer shells [10]. With the specified degree of accuracy the outer layers can be assumed to have no moments the components of the shear strains in them are calculated from approximate formulas that follow from (2.11):

$$\varepsilon_{is}^{(k)}=\varepsilon_{is}+H_{(k)}\chi_{is}+H_i^{(k)}\varepsilon_{s3}+H_s^{(k)}\varepsilon_{i3}+H_i^{(k)}H_s^{(k)}\varepsilon_3$$
$$\left(H_{(k)}=h_{(k)}+\widetilde{t}_{(k)}\approx h_{(k)},\quad H_i^{(k)}=\partial H_{(k)}/\partial\alpha^i\right).$$

Further simplifications of the main relations of the theory of three-layered shells with layers of variable thickness involve, in particular, the introduction of conditions for the slope of the shell [1], limitations on the variability of the thickness of the outer layers or filler [4], etc.

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APPLICATION OF THE METHOD OF INFLUENCE FUNCTIONS IN PROBLEMS OF THE THEORY OF CRACKS FOR ANISOTROPIC PLATES

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Application of analytical methods to estimate the strength of composite materials with cracks and fine inclusions is difficult due to the lack of information concerning the distribution of stresses in a neighborhood of cracktips and inclusions of complex configuration in anisotropic materials. A discussion of this problem and a survey of papers in this direction (mainly for rectilinear cracks and inclusions) can be found, for example, in [1-5].

In what follows, based on the method of influence functions, we present a solution of fundamental problems of planar elasticity theory for anisotropic bodies weakened by curvilinear cuts. Integral representations are constructed, which make it possible to formulate uniformly a solving system of singular integral equations (SIE) for the first, second, and mixed problems of elasticity theory. The effectiveness of the integral representations constructed and of the algorithms presented for numerically solving the resulting SIE is demonstrated by solving a number of problems of crack theory for anisotropic plates.

1. We consider an infinite rectilinear-anisotropic plate weakened by a system of smooth curvilinear nonintersecting cuts $L_j = (a_j, b_j)$, $j = \overline{1, n}$ (Fig. 1). We denote the angle between Ox and the normal n to the left edge of the cut of point $t \in L = \bigcup_{j=1}^{n} L_j$ by $\varphi(t)$. We determine the stress-deformation state (SDS) of such a plate caused by the action of an exterior load $X_n^{\pm}(t) + iY_n^{\pm}(t)$ $(t \in L)$ along the edges of the cuts and by specified stresses at infinity.

Let us assume that the edges of the cuts are not in contact* and the principal vector of the external stresses acting on an edge of the cuts is known. We shall also assume we are given the complex potentials $\Phi_{\nu 0}(z_\nu)$, giving a solution of the problem, for a continuous plate, of external stresses applied at infinity.

*In some problems it is necessary to impose a physical condition, excluding the possibility of an overlapping of the edges of a cut. Such problems are nonlinear and must be solved in an incremental setting, i.e., by stepwise changes of the loading on edges of the cuts.

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